Numerical Optimization

Homework 2

Due 14.05.2014

Give your answers with logical and/or mathematical explanations. Hand-in your homework in the beginning of a lecture on due date. Late submissions will not be accepted.

1. For $f(x) = \frac{1}{2}\|y - Ax\|_2^2$ where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$, and $A \in \mathbb{R}^{k\times n}$, derive the expressions of $\nabla f(x)$ and $\nabla^2 f(x)$ using the chain rule.

2. Compute the gradient $\nabla f(x)$ and the Hessian $\nabla^2 f(x)$ of the Rosenbrock function,

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Show that $x^* = (1, 1)^T$ is the only local minimizer of this function, and that the Hessian matrix at that point is positive definite.

3. Consider $f(x) = x^T Hx$ where $H \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix. Show using the definition of convex functions that $f(x)$ is convex on the domain $\mathbb{R}^n$. Hint: it may be convenient to prove the following equivalent inequality for all $x, y \in \mathbb{R}^n$ and all $\alpha \in [0, 1]$,

$$f(y + \alpha(x - y)) - \alpha f(x) - (1 - \alpha)f(y) \leq 0.$$

4. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Show that the set of global minimizers of f is a convex set.

5. For a square symmetric matrix $A \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$, $x \neq 0$, the quantity

$$\frac{x^T Ax}{x^T x}, \quad x \in \mathbb{R}^n$$

is called the *Rayleigh quotient*. Find a vector x^* that minimizes the Rayleigh quotient for a given A, and the corresponding minimum value. Also, find x^* that maximizes the quotient, and the corresponding maximum value.