Learnability of Description Logic Programs

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Abstract. Carin-$\mathcal{ALN}$ is an interesting new rule learning bias for ILP. By allowing description logic terms as predicates of literals in datalog rules, it extends the normal bias used in ILP as it allows the use of all quantified variables in the body of a clause. It also has at-least and at-most restrictions to access the amount of indeterminism of relations. From a complexity point of view Carin-$\mathcal{ALN}$ allows to handle the difficult indeterminate relations efficiently by abstracting them into determinate aggregations. This paper describes a method which enables the embedding of Carin-$\mathcal{ALN}$ rule subsumption and learning into datalog rule subsumption and learning with numerical constraints. On the theoretical side, this allows us to transfer the learnability results known for ILP to Carin-$\mathcal{ALN}$ rules. On the practical side, this gives us a preprocessing method, which enables ILP systems to learn Carin-$\mathcal{ALN}$ rules just by transforming the data to be analyzed. We show, that this is not only a theoretical result in a first experiment: learning Carin-$\mathcal{ALN}$ rules from a standard ILP dataset.

1 Introduction

Carin was proposed by [Levy and Rouset, 1998] as a combination of the two main approaches to represent and reason about relational knowledge, namely description logic (DL) and first-order horn-logic (HL). In Inductive Logic Programming (ILP) learning first-order horn-logic is investigated in depth, for learning DLs there exist first approaches and theoretical learnability results [Kietz and Morik, 1994; Cohen and Hirsh, 1994; Frazier and Pitt, 1994]. Recently, it was proposed to use Carin-$\mathcal{ALN}$ as a framework for learning [Rouveirol and Ventos, 2000]. This is an interesting extension of ILP as $\mathcal{ALN}$ provides a new bias orthogonal to the one used in ILP, i.e. it allows all quantified descriptions of body-variables, instead of the existential quantified ones in ILP. This allows to handle the difficult indeterminate relations efficiently by abstracting them into a determinate summary. It also has at-least and at-most restrictions, which allow to quantify the amount of indeterminism of these relations. However, up to now there are neither practical nor theoretical results concerning learning the Carin-$\mathcal{ALN}$ language.

This paper is intended to close this gap, by showing how Carin-$\mathcal{ALN}$ learning can be embedded into first-order horn-logic learning as done by ILP-methods. Even, if DL and HL have been shown to be quite incomparable concerning their expressive power [Borgida, 1996] with respect to their usual semantic interpretation of primitive concepts and roles, we show that reasoning in DL can be simulated by reasoning in horn logic with simple numeric constraints as formalized in [Sebag and Rouveirol, 1996] and as present in most ILP-systems, e.g. Foil [Quinlan and Cameron-Jones, 1993] or Progol [Muggleton, 1995]. A simple invertible function encodes normalized concept descriptions into horn clauses using new predicates with an external semantics (as Borgida has shown they are not expressible in horn logic) to represent the DL terms, i.e. from an ILP point of view, learning CARIN-$\mathcal{ALN}$ can be done by learning HL with extended background knowledge that encodes the description logic terms. This encoding not only provides another method to do deductive reasoning, i.e. subsumption, equivalence and satisfiability checking, but also allows ILP methods to learn $\mathcal{ALN}$ concept descriptions and CARIN-$\mathcal{ALN}$ rules. The known border-line of polynomial learnability for ILP can be transfered to CARIN-$\mathcal{ALN}$ as well, as this encoding is a prediction preserving reduction as used in [Cohen, 1995] to obtain PAC-learnability results for ILP.

In section 2 we repeat the basic definitions of the description logic $\mathcal{ALN}$ and define the basis of our new encoding into horn logic. In section 3 we define the description logic program (DLP) formalism. We show how it is related to DLP (3.1) and ILP (3.2) and we define a reasoning procedure by extending the encoding from section 2 to DLP rules (3.3). In section 3.4, we characterize the boarder line of polynomial learnability of DLP rules using the encoding into horn logic. Finally in section 4, we demonstrate with a first experiment, that this encoding can be used to learn DLP rules using a normal ILP-systems on an extended, i.e. preprocessed data-set.

2 The Description Logic $\mathcal{ALN}$

Starting with KL-ONE [Brachman and Schmolze, 1985] an increasing effort has been spent in the development of formal knowledge representation languages to express knowledge about concepts and concept hierarchies. The basic building blocks of description logics are concepts, roles and individuals. Concepts describe the common properties of a collection of individuals and can be considered as unary predicates which are interpreted as sets of objects. Roles are interpreted as binary relations between objects. Each description logic defines also a number of language constructs that can be used to define new concepts and roles. In this paper we use the very basic[1] description logic $\mathcal{ALN}$ under its normal open-world (OWA) semantics, with the language constructs in table 1.

Definition 1 ($\mathcal{ALN}$-terms and their interpretation).
Let P denote a primitive concept, i.e. an unary predicate, and R denote a primitive role, i.e. a binary predicate, n is a positive integer and the C_i are concept terms. The set of all $\mathcal{ALN}$-concept-terms ($\mathcal{C}$) consist of everything in the left

[1] See http://www-db.research.bell-labs.com/user/pfps/papers/krss-spec.ps for further language constructs considered in description logics.

column of the table 1 (and nothing more) and their interpretation (I, Δ) *(see def. 6 below) with respect to the interpretation of primitive concept and roles is defined in the right row of the table 1 (C^I is used as a shortcut for $I(C)$, if C is not primitive). The set of all $\mathcal{ALN}$-role-terms ($\mathcal{R}$) is just the set of all primitive roles R.*

Term (Math)	Term (Classic)	Interpretation
$\top$	**everything**	Δ^I
$\bot$	**nothing**	$\emptyset$
P	P	P^I
$\neg P$	**not** P	$\Delta^I \setminus P^I$
$C_1 \sqcap \ldots \sqcap C_n$	**and** $C_1 \ldots C_n$	$C_1^I \cap \ldots \cap C_n^I$
$\forall R.C$	**all** R C	$\{x \in \Delta^I \mid \forall y \in \Delta^I :< x, y >\in R^I \Rightarrow y \in C^I\}$
$\geq nR$	**at-least** n R	$\{x \in \Delta^I \mid \ \|\{y \in \Delta^I \mid < x, y >\in R^I\}\| \geq n\}$
$\leq nR$	**at-most** n R	$\{x \in \Delta^I \mid \ \|\{y \in \Delta^I \mid < x, y >\in R^I\}\| \leq n\}$

Table 1. Concept terms in $\mathcal{ALN}$ and their model-theoretic interpretation

These language constructs can be used to build complex terms, e.g. the term *train* $\sqcap \forall$ *has_car*.(*car* $\sqcap \leq 0$ *has_load*) can be used to define empty trains (all cars do not have a load), in Michalski's well-known train domain. A statement **not** expressible as a logic program, if we are restricted to the predicates already present in the train domain and cannot use externally computed predicates.

The main reasoning tasks with description logic terms are subsumption, equivalence and satisfiability checking for deduction and the least common subsumer (lcs) for learning.

Definition 2 (subsumption, equivalence, satisfiability and least common subsumer). *The concept description D subsumes the concept description C ($C \sqsubseteq D$) iff $C^I \subseteq D^I$ for all interpretations I; C is satisfiable if there exists an interpretation I such that $C^I \neq \emptyset$; C and D are equivalent ($C \equiv D$) iff $C \sqsubseteq D$ and $D \sqsubseteq C$; and E is the least common subsumer (lcs) of C and D, iff $C \sqsubseteq E$ and $D \sqsubseteq E$ and if there is an E' with $C \sqsubseteq E'$ and $D \sqsubseteq E'$, then $E \sqsubseteq E'$.*

We have chosen $\mathcal{ALN}$, because subsumption checking between $\mathcal{ALN}$ descriptions is polynomial [Donini et al., 1991], and in this paper we always consider an empty terminological component since subsumption checking between $\mathcal{ALN}$ terms with respect to an acyclic terminological component is coNP-complete [Nebel, 1990b] A polynomial time algorithm for the lcs computation of $\mathcal{ALN}$ concepts can be found in [Cohen et al., 1992].

$\mathcal{ALN}$ descriptions are in general not normalized, i.e. there exist several possibilities to express the same concept, e.g. $\bot \equiv (A \sqcap \neg A) \equiv (\geq 2\ R \sqcap \leq 1\ R)$. However, they can be normalized, e.g. by the following set of rewrite rules.

Definition 3 (Normalization of $\mathcal{ALN}$ descriptions). *$norm(C) = C'$ iff $sorted(C) \mapsto \ldots \mapsto C'$ and no rewrite rule is applicable to C', i.e as a first step any conjunction $(C_1 \sqcap \ldots \sqcap C_n)$ is replaced by its sorted equivalent for a total order $<$ that respects $(P_1) < (\neg P_1) < ... < (P_n) < (\neg P_n) < (\geq n_1\ R_1) < (\leq m_1\ R_1) < (\forall R_1.C_1) < ... < (\geq n_n\ R_n) < (\leq m_n\ R_n) < (\forall R_n.C_n)$. Then*

the following rewrite rules are applied to all conjunctions not only the top-level one, as long as possible.
1. $(C_1 \sqcap \ldots \sqcap C_n) \mapsto \bot$, if any $C_i = \bot$
2. $(C_1 \sqcap \ldots \sqcap C_n) \mapsto C_1 \sqcap \ldots \sqcap C_{i-1} \sqcap C_{i+1} \sqcap \ldots \sqcap C_n$, if any $C_i = \top$
3. $(P \sqcap \neg P) \mapsto \bot$
4. $(\leq n\ R \sqcap \geq m\ R) \mapsto \bot$, if $n < m$.
5. $(C \sqcap C) \mapsto C$
6. $(\geq 0R) \mapsto \top$
7. $(\geq n_1\ R \sqcap \geq n_2\ R) \mapsto \geq maximum(n_1, n_2)\ R$
8. $(\leq n_1\ R \sqcap \leq n_2\ R) \mapsto \leq minimum(n_1, n_2)\ R$
9. $(\leq 0\ R \sqcap \forall R.C) \mapsto \leq 0\ R$
10. $(\forall R.\bot) \mapsto \leq 0\ R$
11. $(\forall R.\top) \mapsto \top$
12. $(\forall R.C_1 \sqcap \forall R.C_2) \mapsto \forall R.(merge^2(C_1 \sqcap C_2))$

Lemma 1. *For any two $\mathcal{ALN}$ concept descriptions C_1 and C_2: equivalence ($C_1 \equiv C_2$, iff $norm(C_1) = norm(C_2)$), subsumption ($C \sqsubseteq D$, iff $norm(C \sqcap D) = norm(C)$) and satisfiability ($C$ is satisfiable, iff $norm(C) \neq \bot$) can be computed in $O(n\ log\ n)$, with n being the size of $C \sqcap D$. Proof in [Kietz, 2002]*

The main innovation of this paper is the following encoding of description logic terms into horn clauses. From the logical point of view the new predicates are primitive as all predicates, i.e. they have an external semantic chosen freely by the interpretation function as for any other predicate. The important matter is, that we can prove, that they are always **used** such that semantic constraints (e.g. $I\langle X/\{a_1, \ldots, a_n\}\rangle(cp_P(X)) = 1$, iff $\{a_1, \ldots, a_n\} \subseteq I(P)$) on the interpretation that would formalize the semantic of the encoded DL-terms are respected, i.e that this function is indeed a correct polynomial problem (prediction and deduction preserving) reduction of IDLP to ILP.

Definition 4 ($\mathcal{ALN}$ encoding into constraint horn logic).
$\Phi(C) = h(X) \leftarrow \Phi(norm(C), X)$.
$\Phi(\bot, X) = \bot(X)$
$\Phi(P\ \{\sqcap\ C\}, X) = cp_P(X)\{, \Phi(C, X)\}$
$\Phi(\neg P\ \{\sqcap\ C\}, X) = cn_P(X)\{, \Phi(C, X)\}$
$\Phi(\geq nR \sqcap \leq mR \sqcap \forall R.C_R\ \{\sqcap\ C\}, X) = rr_R(X, [n..m], Y), \Phi(C_R, Y)\{, \Phi(C, X)\}$
$\Phi(\leq mR \sqcap \forall R.C_R\ \{\sqcap\ C\}, X) = rr_R(X, [0..m], Y), \Phi(C_R, Y)\{, \Phi(C, X)\}$
$\Phi(\geq nR \sqcap \forall R.C_R\ \{\sqcap\ C\}, X) = rr_R(X, [n..*], Y), \Phi(C_R, Y)\{, \Phi(C, X)\}$
$\Phi(\geq nR \sqcap \leq mR\ \{\sqcap\ C\}, X) = rr_R(X, [n..m], Y)\{, \bot(Y)\ \textit{if}\ m = 0\}\{, \Phi(C, X)\}$
$\Phi(\forall R.C_R\ \{\sqcap\ C\}, X) = rr_R(X, [0..*], Y), \Phi(C_R, Y)\{, \Phi(C, X)\}$
$\Phi(\leq mR\ \{\sqcap\ C\}, X) = rr_R(X, [0..m], Y)\{, \bot(Y)\ \textit{if}\ m = 0\}\{, \Phi(C, X)\}$
$\Phi(\geq nR\ \{\sqcap\ C\}, X) = rr_R(X, [n..*], Y)\{, \Phi(C, X)\}$
where Y is always a new variable not used so far and $\{\sqcap\ C\}$ means, if there

[2] Merging two sorted lists into one sorted list as in merge-sort. In an optimized implementation the recursive application of the normalisation rules should be integrated into that linear (in the size of the conjunctions) process.

are conjuncts left, recursion on them $\{, \Phi(C,X)\}$ *has to continue. For any normalised concept term only the first matching one has to by applied.*

Let us illustrate our encoding function on our train example: $\Phi(\mathit{train} \sqcap \forall\ \mathit{has\_car}.(\mathit{car} \sqcap \leq 0\ \mathit{has\_load})) = h(X) \leftarrow \mathit{cp\_train}(X)$, $\mathit{rr\_has\_car}(X,[0..*],Y)$, $\mathit{cp\_car}(Y)$, $\mathit{rr\_has\_load}(Y,[0..0],Z)$, $\bot(Z)$. Note, that this clause has a very different meaning than the clause $h(X) \leftarrow \mathit{train}(X)$, $\mathit{has\_car}(X,Y)$, $\mathit{car}(Y)$, $\mathit{has\_load}(Y,Z)$. The first one must be interpreted as being true for every **set of empty trains**, i.e. X, Y and Z are variables over set of terms. The second one is true for every **single train with has a car, which has a load**. The predicates (and variables) in the first clauses are meta-predicates (set-variables) over the ones in the second clause (individual variables), e.g. like `findall` in Prolog with a specific call using predicates of the second clause. This difference becomes especially important in our DLP language, i.e. in the next section, where both kinds of literals can occur in a single clause.

Nevertheless, we are now nearly able to simulate DL subsumption with θ-subsumption and lcs with lgg. There are only two very small and easy extensions of θ-subsumption and lgg (to $\theta_{I\bot}$-subsumption and $lgg_{I\bot}$) needed:

The handling of sub-terms representing intervals of numbers, e.g. a term like $[0..*]$ should θ_I-subsume a term like $[1..5]$. More precisely an interval $[Min_1..Max_1]$ θ_I-subsumes an interval $[Min_2..Max_2]$, iff $Min_1 \leq Min_2$ and $Max_2 \leq Max_1$; and the lgg_I of two intervals $[Min_1..Max_1]$ and $[Min_2..Max_2]$ is the interval $[\mathit{minimum}(Min_1, Min_2)..\mathit{maximum}(Max_1, Max_2)]$[3]. The handling of nothing, i.e. $\bot(X)$ should be $\theta_\bot$-subsumed by $\Phi(C,X)$ for any concept description C, e.g. by any sub-clause containing the relation-chains starting with X; and the $lgg_\bot$ of $\bot(X)$ and $\Phi(C,X)$ is $\Phi(C,X)$. This does not really need an extension of the logic, it can be simulated with normal θ-subsumption and the following sub-clause for bottom: for any primitive role R: $rr_R(\mathit{bottom},[*..0],\mathit{bottom})$ and for any primitive concept C: $\mathit{cp}_C(\mathit{bottom})$, $\mathit{cn}_c(\mathit{bottom})$. Every $\Phi(C,X)$ θ-subsumes this cyclic structure with a θ that maps every variable in it to the term *bottom* and the lgg of every $\Phi(C,X)$ and that sub-clause is $\Phi(C,X)$, e.g. $lgg_I(rr_R(\mathit{bottom},[*..0],\mathit{bottom}), rr_R(X,[Min..Max],Y)) = rr_R(X,[Min..Max],Y)$ and everything present in $\Phi(C,X)$ has a selection [Plotkin, 1970] within the lgg.

Theorem 1 (Simulation of subsumption and lcs). *A concept description* C *subsumes a concept description* D *(*$D \sqsubseteq C$*), if and only if the encoding of* C $\theta_{I\bot}$*-subsumes the encoding of* D *(*$\Phi(C) \vdash_{\theta_{I\bot}} \Phi(D)$*), and* $lcs(C,D) \equiv \Phi^{-1}(lgg_{I\bot}(\Phi(C),\Phi(D)))$

[3] Formally this corresponds to reasoning and learning with simple numeric constraints as present in Constraint Logic Programs (CLP). In in most ILP-systems, e.g. Foil [Quinlan and Cameron-Jones, 1993] or Progol [Muggleton, 1995] this is done with the help of the computed built-in predicates $\leq$ and/or $\geq$. For Foil or Progol a more suitable encoding is $rr_R(X, n_{atleast}, m_{atmost}, Y)$, as they can learn c_{min} and c_{max} in literals like $c_{min} \leq n_{atleast}$ and $m_{atmost} \leq c_{max}$. [Sebag and Rouveirol, 1996] have analysed this CLP extension of ILP systems and it is easy to see that the properties of θ-subsumption also hold for θ_I-subsumption.

This theorem directly follows from the similarity between the encoding Φ of the normalized concept terms and $\theta_{I\perp}$-subsumption and the correctness of the structural subsumption algorithm given in [Cohen et al., 1992]. There are some more nice properties for learning. Any clause which subsumes a set of such clauses, does so deterministically [Kietz and Lübbe, 1994], i.e. is determinate [Muggleton and Feng, 1992], as any $rr_R(X, I, Y)$ occurs just once for any variable X. The relation chains in the clause (i.e. the chains of $rr_R(X, I, Y)$ literals) are not only acyclic but even tree-structured (as the DL-Term is a tree), i.e. subsumption (coverage test), learning, and propositionalisation are very easy, e.g. the depth limit i needed for the polynomial learnability of (cyclic) ij-determinate clauses is not needed. Φ is a bijective, invertible function, i.e $\Phi^{-1}(\Phi(C)) \equiv C$, i.e. it allows to decode generalized (i.e. learned) clauses: If D is a linked clause[4], and $D \vdash_{\theta_{I\perp}} \Phi(C)$ for any $\mathcal{ALN}$ description C, then $\Phi^{-1}(D)$ is totally invertible and produces a valid description logic term.

3 Induction of Description Logic Programs

CARIN as proposed in [Levy and Rouset, 1998] combines first-order function-free horn logic with description logic by allowing description logic terms as body literals in horn rules. Concept terms represent unary predicates and role-terms represent binary predicates. The direct use of primitive concepts and roles is indistinguishable from the use of unary and binary predicates in FOL, but the use of concept terms, i.e. descriptions contain all, at-least and at-most adds expressive power to the language. Here is an example of a CARIN-$\mathcal{ALN}$ rule using this expressive power. Note, that this cannot be expressed in neither horn logic nor $\mathcal{ALN}$ alone. We bracket DL-terms with [], wherever we think that this will increase readability.

$$\begin{aligned} &\mathit{east\_train}(X) \leftarrow \mathit{has\_car}(X,Y),\ \mathit{has\_car}(X,Z), \mathit{same\_shape}(Y,Z), \\ &\qquad [\mathit{train} \sqcap \leq 2\ \mathit{has\_car} \sqcap \forall\ \mathit{has\_car}.[\mathit{car} \sqcap \leq 0\ \mathit{has\_load}]](X). \end{aligned}$$

The definition of our DLP language is based on datalog, i.e. function-free logic programs. The important extension is that we allow $\mathcal{ALN}$-concept-terms as one-place predicates and $\mathcal{ALN}$-role-terms as two-place-predicates.

Definition 5 (The DLP language). *Variables $\mathcal{V}$, constants $\mathcal{F}$, and terms $\mathcal{T}$ are as usual in datalog. Let $\mathcal{P} = (\mathcal{P}_n)_{n\in\mathbb{N}}$ be a signature of predicate symbols, i.e. for $n \in \mathbb{N}$ is $p \in \mathcal{P}_n$ a predicatssymbols of arity n, e.g. $p, q, r \in \mathcal{P}_n$. The extension to description logic programms extends $\mathcal{P}_1$ by the set of all concept-terms $\mathcal{C}$ and $\mathcal{P}_2$ by the set of all role-terms $\mathcal{R}$ (see Definition 1). Basesd on that atoms AT, literals LIT, clauses, facts, rules and desricption logic programs are again defined as in datalog.*

[4] There is no partition of literals such that one partition does not share at-least one variable with any other partition

Definition 6 (Interpretations and Models of clause sets and description logic programs).
Let Δ be a domain and I an interpretation function mapping clauses to truth values (0 and 1), n-ary predicates ($P_n \in \mathcal{P}$) to n-ary relations (written P_n^I) over the domain Δ, terms ($t \in \mathcal{F}$) to elements of Δ (written t^I), $\mathcal{ALN}$-concept-terms ($C \in \mathcal{C}$)as defined in def. 1 and $\mathcal{ALN}$-role-terms ($R \in \mathcal{R}$)as defined in def. 1. An interpretation (I, Δ) satisfies (is a model of) a clause set, iff every clause is satisfied by (I, Δ). A clause is satisfied by (I, Δ), iff the interpretation function assigns the truth value 1 to it. For all interpretations (I, Δ) the following must hold [5]*:*

1. $\forall t_1, t_2 \in \mathcal{F} : t_1 \neq t_2$ *iff* $t_1^I \neq t_2^I$,[6]
2. $I(\Box) = 0$, *and* $\forall X \in \mathcal{T} : I(\bot(X)) = 0$
3. $I(\{l_1, \ldots, l_n\}) = 1$, *iff for all vectors* $\boldsymbol{a} = a_1, \ldots, a_m$, $a_i \in \Delta, (1 \leq i \leq m) : I\langle X_1/a_1, \ldots, X_m/a_m\rangle(l_1) = 1$ *or ... or* $I\langle X_1/a_1, \ldots, X_m/a_m\rangle(l_n) = 1$, *where* $\boldsymbol{X} = X_1, \ldots, X_m$ *is the vector of all variables in the clause* $\{l_1, \ldots, l_n\}$,
4. $I(\exists\{l_1, \ldots, l_n\})^7 = 1$, *iff there exists a vector* $\boldsymbol{a} = a_1, \ldots, a_m$, $a_i \in \Delta, (1 \leq i \leq m) : I\langle X_1/a_1, \ldots, X_m/a_m\rangle(l_1) = 1$ *or ... or* $I\langle X_1/a_1, \ldots, X_m/a_m\rangle(l_n) = 1$, *where* $\boldsymbol{X} = X_1, \ldots, X_m$ *is the vector of all variables in the clause* $\{l_1, \ldots, l_n\}$,
5. $I(\neg A) = 1$ *iff* $I(A) = 0$
6. $I(P_n(t_1, \ldots, t_n)) = 1$ *iff* $t_1^I, \ldots, t_n^I \in P_n^I$
7. $I(C(t)) = 1$ *iff* $t^I \in C^I$, *with* C^I *defined in table 1*
8. $I(R(t_1, t_2)) = 1$ *iff* $t_1^I, t_2^I \in R^I$.

Definition 7 (logical consequence).
A clause C follows from (is a logical consequence of) a clause set P (written $P \models C$), iff all models of P are also models of C. A clause set P_1 follows from (is a logical consequence of) a clause set P_2 (written $P_2 \models P_1$), iff all models of P_2 are also models of P_1.

We also use the normal ILP definition of learning to define learning of description logic programs (called Induction of Description Logic Programs or short IDLP).

Definition 8 (The IDLP Learning Problem).
Given a logical language $\mathcal{L}$ (i.e. our language DLP) with a consequence relation $\models$, background knowledge B in a Language $LB \subseteq \mathcal{L}$, positive and negative examples $E = E^+ \cup E^-$ in a language $LE \subseteq \mathcal{L}$ consistent with B ($B, E \not\models \Box$) and

[5] $I\langle X/a\rangle(l)$ means that during the interpretation $I(l)$ the variable X in the literal l is always interpreted as the domain element a. It must not be confused with a substitution as a domain element a may not have a constant in $\mathcal{F}$ denoting it, i.e. $I^{-1}(a)$ may be undefined.

[6] In normal first-order logic such a unique name assumption for constants is usually not used, but for counting in DLs with number-restrictions it is very useful. It must not be confused with the object identity restriction on variables (substitutions must be injective) also used in some ILP approaches.

[7] This formula is not an expression **in** our language, but we need to interpret it in a lemma **about** our language.

not already a consequence of B ($\forall e \in E : B \not\models e$), and a hypothesis language $LH \subseteq \mathcal{L}$. Find a hypothesis $h \in LH$ such that:

(I) ($B, h, E \not\models \Box$), i.e. h is consistent with B and E.
(II) ($B, h \models E^+$), i.e. h and B explain E^+.
(III) ($B, h \not\models E^-$), i.e. h and B do not explain E^-.

The tuple $(\models, LB, LE, LH)$ *is called the* IDLP learning problem*. Deciding whether there exists such an $h \in LH$, is called the* IDLP consistency problem*. An algorithm which accepts any $B \in LB$ and $E \in LE$ as input and computes such an $h \in LH$ if it exists, or "no" if it does not exist is called an* IDLP learning-algorithm*.*

As we are interested in polynomial learnability results [Cohen, 1995] for DLP, we are especially interested in IDLP problems, where B and E consist of ground (variable-free) facts, and H is restricted to 1-clause Programs (see section 3.4).

3.1 The relation between (learning) Carin-$\mathcal{ALN}$ and (I)DLP

At first this looks quite different from the definition of Carin-$\mathcal{ALN}$ in [Levy and Rouset, 1998] or the Carin-$\mathcal{ALN}$ learning problem as defined in [Rouveirol and Ventos, 2000], but it is in fact quite similar. We of course do not claim equivalence of the formalisms, but we think that we have transported the important (to us) ideas from Carin-$\mathcal{ALN}$ into a more (inductive) logic programming oriented form. The DLP formalism defined above is in general to expressive compared with Carin-$\mathcal{ALN}$, as so far we have neither forbidden recursion nor DL-literals as the head of rules, two things known to be quite difficult to reason with in Carin-$\mathcal{ALN}$ [Levy and Rouset, 1998]. But we have to introduce such restrictions as well, as rules like $connected(X, Y) \leftarrow node(X), node(Y), [\leq 0\ connected](X)$ expressible in the formalism are difficult to interpret[8] under the normal model-theoretic semantics we used. But as in ILP we want to start with the general idea, and then introduce the restrictions needed to come to a characterisation of polynomial learnability in the end. Given the goal of learning DLPs, recursive description logic programs are not very interesting anyway, as the set of know polynomial learnable recursive logic programs is very restricted and so far only of interest for synthesis of toy programs and not for real data analysis or mining applications.

The other main difference between DLP and Carin-$\mathcal{ALN}$ is that we have unified assertional component (the set of facts from our DLP) and rule component (the set of rules from our DLP) of Carin-$\mathcal{ALN}$, and - as already discussed in section 2 - we have dropped the terminological component of Carin-$\mathcal{ALN}$, but are instead able to express rules and assertions, where concepts are already expanded with respect to the (acyclic) terminological axioms [Nebel, 1990a], e.g.

[8] They do not have a (stable) model, so they have to be interpreted as contradictory, but an interpretation as default rule may be more adequate. But this is not a topic of this paper.

instead of separating terminological axioms T, rules R and assertions A as in the following example from [Rouveirol and Ventos, 2000]:

$$T = \{empty\_car \equiv car \sqcap \leq 0\ has\_load,$$
$$empty\_train \equiv train \sqcap \forall has\_car.empty\_car\}$$
$$R = \{east\_train(X) \leftarrow empty\_train(X)\}$$
$$A = \{train(a),\ has\_car(a,d),\ \leq 1\ has\_car(a),\ empty\_car(d)\}$$

we require the representation of the equivalent (with respect to assertional and rule consequences) terminological expanded rule and facts within one description logic program: $train(a)$. $has\_car(a,d)$. $[\leq 1 has\_car](a)$. $[car \sqcap \leq 0\ has\_load](d)$. $east\_train(X) \leftarrow [train \sqcap \forall has\_car.[car \sqcap \leq 0\ has\_load]](X)$.

Except for the dropped $\mathcal{ALN}$ terminological component which makes polynomial learnability results possible (with a terminological component learning is at-least as hard as reasoning, i.e coNP-hard [Nebel, 1990b]) this is only a syntactic difference.

Concerning the learning framework in [Rouveirol and Ventos, 2000] and the IDLP problem, the main difference is that they use the learning from interpretation setting, whereas we use the normal ILP setting. For non-recursive clauses, this makes no difference, and they could be easily transformed into each other with only a polynomial size difference, if B and E consist of ground facts, i.e. take the whole B as interpretation for each example, or take the union of all interpretations as B.

3.2 The relation between (I)LP and (I)DLP

As known from Carin-$\mathcal{ALN}$, there are also serious differences between DLP and LP. A set of rules may have more consequences than the union of the consequences of the individual rules, e.g. let
$P_1 = \{train(a).\ east\_train(X) \leftarrow [train \sqcap \leq 1\ has\_car](X).\}$ and
$P_2 = \{train(a).\ east\_train(X) \leftarrow [train \sqcap \geq 1\ has\_car](X).\}$.
Neither $P_1 \models east\_train(a)$ nor $P_2 \models east\_train(a)$ but $P_1 \cup P_2 \models east\_train(a)$ as $[\geq 1\ has\_car](X) \vee [\leq 1\ has\_car](X)$ is a tautology.

Theoretically (section 3.4), this does not hinder our transfer of learnability results from LP to DLP very much as most results concerning polynomial learnability of ILP are restricted to one rule programs [Cohen, 1995], but this is a serious problem for applying these positive DLP results practically as done in ILP. Most ILP systems use the greedy strategy of learning rules individually such that no negative examples are covered by any rule until all positive examples are covered by a rule. This difference between DLP and LP means that a rule set may cover negative examples, even if no rule does. However, this is only a problem in learning DLPs from DLP facts, but not in the practically much more important subproblem of learning DLPs from ILP data sets, e.g. from relational databases. Under the usual open world assumption in DLP no rule ever learned from an ILP data set will contain an at-most or all restriction, as under the open-world assumption (OWA) a rule with an at-most or all restriction will never cover an example described by datalog facts only, e.g. P_1 will never cover a train as without an explicit at-most restriction in the examples, i.e. the OWA means we will never know that we know all cars of a train. This however makes learning DLP quite useless as well, as only at-least restrictions are learnable from ILP facts, but neither at-most nor all restriction.

There is an easy practical way out of this problem used in section 4 to learn DLP programs from ILP data sets namely assuming that the background knowledge contains all relevant background knowledge for the examples (a closed world assumption on B). In that case we can also learn at-most and all restriction and it is always possible to decide which side of a tautology is applicable, e.g. if we know that we know all the cars of the trains under consideration, we can decide by counting, if rule P_1 or P_2 or both are applicable. Also for a primitive concept P we are able to decide, if P or $\neg P$ is true. In summary, we do not have to worry about this reasoning problem with respect to the goals of this paper. Theoretically we only need reasoning with one clause programs, practically we need a closed world assumption to learn DLPs and in both cases this problem disappears.

Another difference between LP and DLP reasoning is that a DLP like the one above also contains implicit knowledge not only due to rule reasoning as usual in logic programing but also due to interactions between facts with description logic literals (DL-literals) and normal literals (HL-literals[9]), i.e. the fact $[\forall has\_car.[car \sqcap \leq 0\ has\_load]](a)$ is true in every model of the DLP as a consequence of the interaction between all the above facts and after we have deduced that $east\_train(a)$ can be deduced as a consequence of the facts and the rule.

3.3 Subsumption between DLP clauses

We have to make such implicit literals explicit, as a reasoning procedure based on substructure matching like subsumption would be incomplete otherwise, e.g. $h(X) \leftarrow a(X,Y) \Leftrightarrow h(X) \leftarrow (\geq 1a)(X)$, but $h(X) \leftarrow a(X,Y) \not\vdash_{\theta_{I\perp}} h(X) \leftarrow \Phi(\geq 1a, X)$ $h(X) \leftarrow \Phi(\geq 1a, X) \not\vdash_{\theta_{I\perp}} h(X) \leftarrow a(X,Y)$.

If the rules were ground, the interaction between HL- and DL-terms would correspond to what is called ABox reasoning in description logic, i.e. inferring HL-literals corresponds to ABox completion [Baader and Sattler, 2000] and inferring DL-literals corresponds to computing the most specific concept [Baader and Küsters, 1998] for every variable. Goasdoué, Rouveirol and Ventos [2001] have put them together as completion rules to formalize example saturation for learning under OI-subsumption. The adaptation of their rules to θ-subsumption lead to the following completion rules to be applied to a set of DLP facts, i.e. either background knowledge B or the body B of a DLP rule $H \leftarrow B$.

Definition 9 (Disjointness Clique). *A set of terms $\{X_1, ..., X_n\}$ is called a disjointness clique of size n of a rule body B, iff for all pairs $\{X_i, X_j\} \subseteq$*

[9] Primitive roles and concepts strictly belong to both classes, but we will reference and handle primitive roles and primitive concepts as HL-literals

$\{X_1, \ldots, X_n\}$, with $i \neq j$ either X_i and X_j are not unifiable terms, i.e. different constants, or $\{C_i(X_i), C_j(X_j)\} \subseteq B$ and $C_i \sqcap C_j \equiv \bot$, i.e. there must be no model, where they are interpreted as the same individual.

Definition 10 (Completion for DLP under θ-Subsumption). *Let B a set of DLP facts.*

1. **apply at-least restriction:** *if there exists a substitution σ such that $((\geq n\ r)(X_0))\sigma \in B$ for $n \geq 1$ and $(\{r(X_0, X_1)\})\sigma \notin B$ then $B \rightharpoonup B \cup (\{r(X_0, U)\})\sigma$, and U is a new variable not occurring in B (and σ) so far.*
2. **apply value restriction:** *if there exists a substitution σ such that $(\{(\forall r.C)(X_0), r(X_0, X_1)\})\sigma \subseteq B$ and $C(X_1)\sigma \notin B$ then $B \rightharpoonup B \cup \{C(X_1)\sigma\}$.*
3. **infer at-least restriction:** *If there exists a substitution σ such that*
   - *$(\{r(X_0, X_1), \ldots, r(X_0, X_n)\})\sigma \subseteq B$, and there is no further $r(X_0, X_{n+1})\sigma$ in B, and*
   - *k is size of the maximal disjointness clique of $\{X_1, \ldots, X_n\}\sigma$ in B, and*
   - *$((\geq m\ r)(X_0))\sigma \notin B$ for any $m \geq k$*

   then $B \rightharpoonup B \cup \{[\geq k\ r](X_0)\}\sigma$.
4. **infer value restriction:** *If there exists a substitution σ such that $(\{(\leq n\ r)(X_0), r(X_0, X_1), \ldots, r(X_0, X_n), C_1(X_1), \ldots, C_n(X_n)\})\sigma \subseteq B$, and $\{X_1, ..., X_n\}\sigma$ is a disjointness clique is of size n, let $C = lcs(C_1, \ldots, C_n)$ and $((\forall r.C)(X_0))\sigma \notin B$ then $B \rightharpoonup B \cup \{[\forall r.C](X_0)\}\sigma$,*
5. **collect and normalize concept-terms over the same variable:** *if $C(X_0)\sigma \in B$ and $D(X_0)\sigma \in B$ with C and D are DL-terms then $B \rightharpoonup B \setminus \{C(X_0)\sigma, D(X_0)\sigma\} \cup \{[norm(C \sqcap D)](X_0)\}\sigma$*

This corresponds to ABox reasoning presented as equivalent to tableux-based terminological component reasoning in [Baader and Sattler, 2000], but for learning a tableux reasoning approach which does only consistency checking is not sufficient. We need a constructive normalisation approach to have the consequences available as input for learning. This of course is only feasible for a simple DL like $\mathcal{ALN}$. It in fact corresponds to inferring for each term t occurring in B a depth-bounded acyclic approximation of it's (cyclic) most specific concept as done for $\mathcal{ALN}$-ABoxes in [Baader and Küsters, 1998].

Lemma 2 (The completion rules are correct). *For any B, if $B \rightharpoonup B'$, then $\exists B \models \exists B'$. Proof in [Kietz, 2002]*

Lemma 3 (The completion rules are complete). *For all atoms $a \in AT$, whenever B_0 is consistent ($\exists B_0 \not\models \exists\bot(X)$), if $\exists B_0 \models \exists a$, there exits a chain of $B_0 \rightharpoonup B_1 \rightharpoonup \ldots \rightharpoonup B_n$ such that for some substitution σ either $a\sigma \in B_n\sigma$ or $a\sigma = C_1(t)$ and there exist a $C_2(t)\sigma \in B_n\sigma$ and $C_2 \sqsubseteq C_1$. Proof in [Kietz, 2002]*

Theorem 2 (Subsumption between completed encoded rules is sound). *Let $depth(C)$ the depth of the deepest $\forall$ nesting in C, let $\varphi(D, i)$ denote the completion of D up to depth i, i.e. the application of the rules 1-5 as often as possible, without generating a $\forall$ nesting deeper than i and Φ_r the extension of Φ to rules, such that all DL-literals $DL(X)$ in the rule are encoded with*

$\Phi(norm(DL), X)$ and everything else is returned unchanged. A non-recursive DLP rule C implies a non-tautological DLP rule D ($C \models D$), if and only if the encoding Φ_r of C $\theta_{I\perp}$-subsumes the encoding Φ_r of the completion φ of D ($\Phi_r(C) \vdash_{\theta_{I\perp}} \Phi_r(\varphi(D, depth(C)))$). Proof in [Kietz, 2002]

Theorem 3 (Completion is polynomial for depth-bounded DL terms). *Proof in [Kietz, 2002]*

As Φ and φ are polynomial for ground clauses, the complexity of $C \models D$ is that of θ-subsumption, if C is ground. From [Kietz and Lübbe, 1994] we know, that θ-subsumption is NP-complete in general and polynomial for an arbitrary horn clause D, if $C = C_0 \leftarrow C_{DET}, LOC_1, \ldots, LOC_n$ is a horn clause where $C_0 \leftarrow C_{DET}$ is determinate[10] with respect to D and each $LOC_i, 1 \leq i \leq n$, is a k–local[11]. In that case, $C \vdash_\theta D$ can be decided with $O(\|C\| * \|C_{DET}\| * \|D\| + \|LOC_1, \ldots, LOC_n\|^2 + n * (k^k * \|D\|))$ unification attempts.

Theorem 4 (Simulation of lgg). *Let $E = \Phi_r^{-1} lgg_{I\perp}(\Phi_r(\varphi(C, i)), \Phi_r(\varphi(D, i)))$. E is the least general generalization (lgg) with depth at-most i of C and D, i.e. $E \models C$ and $E \models D$ and if there is an E' with depth at most i with $E' \models C$ and $E' \models D$, then $E' \models E$.*[12]

3.4 Learning Carin-$\mathcal{ALN}$

With the definition of learning and theorem 2 we can immediately conclude on the learnability of DLP programs consisting of a single rule.

Corollary 1. *A IDLP rule learning problem ($\models$,ground DLP facts, ground HL facts, $1-$clause DLP programms) is polynomially learnable, if and only if the corresponding ILP learning problem ($\models$, $\Phi_r(\varphi$(ground DLP facts,i)), ground HL facts, $\Phi_r(\varphi(1-$clause DLP programms,i))) is polynomial learnable.*

As the boarder line of polynomial learnability is quite well known for ILP due to a lot of positive and negative learnability results (see [Kietz, 1996; Kietz and Džeroski, 1994; Cohen and Page, 1995] for overviews), we are now able to characterize it for DLP rules as well. One of the most expressive (single horn clause) ILP-problem that is polynomial learnable, is learning a i, j-determinate-k-literal-local horn clause (see [Kietz and Lübbe, 1994] for definitions and discussions of these restrictions).

Theorem 5. *A 1-clause DLP program, where the HL part is i, j-determinate-k-literal-local and DL-terms are depth-bounded by i and are only allowed on head or determinate variables is polynomial learnable*[13]*.*

[10] The n-ary generalisation of what is called an attribute in DL, i.e. there is a determinate way to find a unique match of the output variables.

[11] It does not share variables with another local, which not also occur in C_0 or C_{DET} and it has at-most k literals (k–literal local) or at-most k variables (k–variable local)

[12] Without depth limit the lgg may not exist, i.e. is infinite due to rule 4.

[13] This includes that they are PAC learnable and polynomial predictable, as Φ is a polynomial prediction preserving reduction.

Proof. The encoding of such a clause is a $2*i, j$-determinate-k-literal-local horn clause as the DL-term encodings are in itself determinate, start on a determinate variables and are bound in depth by i as well and as such they are polynomial learnable [Cohen, 1995]. □

4 Applying the results

This encoding does not only produce theoretical results it also works in practise. Let us demonstrate this with the ILP dataset (available from MLNet) for Mesh-Design [Džeroski and Dolsak, 1992].

4.1 Learning from databases using closed world assumption

As discussed in section 3.2, datalog facts or databases are not adequate to learn DLPs under OWA and the CWA is quite natural a way out for learning [Helft, 1989] as well as for databases. We do not want to close the world totally, but just locally, i.e we assume that if an object is described at all it is described completely, but we don't want to assume that we know about all objects of any world. The following two non-monotonic rules are doing just that.

6. **infer at-most restriction:** If there exists a substitution σ, such that
   - $(\{r(X_0, X_1), \ldots, r(X_0, X_n)\})\sigma \subseteq B$, and there is no further $r(X_0, X_{n+1})\sigma$ in B, and
   - k is size of the maximal disjointness clique of $\{X_1, \ldots, X_n\}\sigma$ in B, and
   - there is no $[\leq m\ r](X_0)\sigma$ in B such that $m \leq k$

   then $B := B \cup \{[\leq m\ r](X_0)\}\sigma$.
7. **infer concept negation:** Let c be a constant appearing in some literal of B, and P be any primitive concept, if $P(c) \notin B$, then $B := B \cup \{\neg P(c)\}$.

Given N constants (or variables) in the rule, and M primitive concepts, only $N*M$ additional literals are introduced at the literal level by the local CWA rule 7, i.e. only a polynomial amount. An $\forall r.\neg P$ can only be added, by rule 4 out of the introduced literals. But $\forall r.\neg P$ does not follow for every role under CWA, i.e under CWA only the literal $\neg P(b)$ is added by rule 7 to $R(a,b), R(a,c), P(c)$, but $(\forall r.\neg P)(a)$ is obviously not true and not added. Theorem 2 proves polynomial complexity in the size of the input of rule 4 under OWA. Therefore the complexity under CWA is polynomial as well.

4.2 Learning Mesh-Design in DLP

We have chosen Mesh-Design as it only contains unary and binary predicates, i.e. perfectly fits description logic without the need for further preprocessing. We have chosen Cilgg [Kietz, 1996] as the ILP-systems to learn the encoding descriptions as it has interval handling, is optimized for determinate literals (e.g. like Golem [Muggleton and Feng, 1992]) and is able to learn k-literal-local clauses completely for small k. As depth limit we had chosen 1 for HL-literals. In

```
mesh(A,2):- [usual ⊓
             ≥ 2 neighbour ⊓
             ≤ 3 neighbour ⊓
             ∀ neighbour.not_loaded
             ](A).
```

```
mesh(A,2):- [usual ⊓
             ≥ 1 opposite ⊓
             ≤ 1 opposite ⊓
             ∀ opposite.[fixed ⊓
                         ≥ 2 opposite ⊓
                         ≤ 2 opposite
                         ]
             ](A).
```

Fig. 1. Two of the description logic rules learned for Mesh-Design

Mesh-Design this produces 4-literal-local ground starting (bottom) clauses. We restricted DL-terms to the head variable/object as this is the only determinate one and to the depth of 3, as deeper terms are very difficult to understand. We have made three experiments, one with only the literals generated from the DL-term, one with only the HL-literals, and one with both (CL). Cilgg [Kietz, 1996] has two possibilities to use the learned rules for testing, the normal deductive one for most general discriminations (generated similar as in Golem from the learned lggs), and a k-nearest neighbor classification (20-NN in this case) using the learned lggs. The results are in table 2 together with the results reported in [Džeroski and Dolsak, 1992] and Indigo [personal note from P. Geibel, 1996]. The table gives the number of correctly (and uniquely) classified edges per object using rules learned from the other objects. The results indicates that the extended language helps to learn better rules in Mesh-Design. The used runtime also reflects the theoretical result that learning the DL-part is easier than learning the HL-part. However, this single experiment is not sufficient to claim the usefulness of Carin-$\mathcal{ALN}$ as a hypothesis language in general. But it clearly shows, that this gain in expressivity of the language can help to achive better results, and that the price to pay for this gain in terms of complexity and computational costs is not very high, i.e. it enables normal ILP-system to learn rules like the ones in Figure 1 efficiently, which are not in the normal ILP bias and which may be useful in some application domains.

	A	B	C	D	E	Σ	%	Avg. CPU Time
Maximum	52	38	28	57	89	268		
Default	9	9	6	13	23	60	22	
Foil	17	5	7	9	5	43	16	(5m, in 1992)
mFoil	22	12	9	6	10	59	22	(2h, in 1992)
Golem	17	9	5	11	10	52	20	(1h, in 1992)
Indigo	21	14	9	18	33	95	36	(9h, in 1996)
Claudien	31	9	5	19	15	79	30	(16m, in 1992)
Cilgg(1996)	19	16	6	10	9	60	22	(85s, in 1996)
Cilgg DL	16	8	5	10	12	51	19	8s
Cilgg HL	22	14	8	13	5	62	23	11s
Cilgg CL	19	16	7	14	23	79	30	22s
Cilgg 20-NN DL	20	12	9	16	38	95	36	19s
Cilgg 20-NN HL	17	14	9	18	41	99	38	23s
Cilgg 20-NN CL	26	12	10	18	37	103	39	52s

Table 2. Comparison of ILP-Approaches learning the MESH-Data set

5 Summary and Outlook

We have characterizes the boarder line of polynomial learnability of CARIN-$\mathcal{ALN}$ rules from ground facts by reducing it to the well investigated boarder-line of polynomial learnability in ILP. This work should be extended to more expressive forms of background knowledge, e.g. terminological axioms (an ontology) and description logic programs. We also showed in a first experiment, that this theoretical encoding is applicable in practice. However, careful experimentation about the usefulness of using CARIN-$\mathcal{ALN}$ as hypothesis language is still missing. But, we provided a data preprocessing method, that allows us to do that with a broad range of ILP-systems on a broad range of ILP-applications. On the theoretical side the IDLP framework could serve to analyse the learnability of the related new aggregation-based ILP approaches [Krogel and Wrobel, 2001] .

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